Gauge Symmetry and its Implications for the Schwinger-Dyson Equations

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Abstract

Gauge theories have been a corner stone of the description of the world at the level of fundamental particles. The Lagrangian or the action describing the corresponding interactions is invariant under certain gauge transformations and contains the dependence on a covariant gauge parameter. This symmetry is reflected in terms of the Ward-Green-Takahashi (or the Slavnov-Taylor) identities (WGTI) which relate various Green functions among each other, and the Landau-Khalatnikov-Fradkin transformations (LKFT) which relate a Green function in a particular gauge to it in an arbitrary covariant one. As an outcome, all physical observables should be independent of the choice of gauge. The most systematic scheme to solve quantum field theories (QFT) is perturbation theory where the above-mentioned identities are satisfied at every order of approximation. This feature is exploited highly usefully as a verification of the results obtained after time- and effort-consuming exercises.

As it stands, not all natural phenomena are realized in the perturbative regime of QFTs. Therefore, one is inevitably led to make efforts to solve these theories in a non perturbative fashion. A natural starting point for such studies in continuum are Schwinger-Dyson Equations (SDEs). One of the most undesirable features associated with their non-perturbative truncations is the loss of highly and rightly esteemed gauge invariance if special care is not taken. In this work, we present a summary of the modern approach to recover gauge invariance in a systematic manner invoking not only WGTI and LKFT but also requiring the truncation schemes to match onto the blindly trustable perturbative ones at every order of approximation. It is a highly difficult task but not impossible.

1 Introduction

In the spectacular realm of quantum fields, gauge theories stand out as they lie at the heart of our quest to unveil the nature of hidden and mysterious forces which orchestrate the dance of events in the backdrop of the flow of time. Whether it be electromagnetic forces which make a close contact with direct human experience, strong forces which hold the nuclei intact or weak forces which are manifest in radioactive decays of elements, gauge theories, namely, quantum electrodynamics (QED), chromodynamics (QCD) and weak dynamics (QWD), seem to provide their

accurate description. Among these, quantum electrodynamics enjoys a special place as the quantitative precision of its predictions in the perturbative domain is outstandingly high. Electromagnetic and weak forces can be wedded harmonically into the celebrated Standard Model of particle physics put forward by Salam, Weinberg and Glashow in late 60's. Its tremendous success in collating experimental results has frustrated the scientific community for years which is seeking physics beyond it. Keeping gravity aloof, in simple extensions of the standard model, QCD can also be potentially unified with the other two.

The infallible proof of the harmonic co-existence of experiment and theory for the above-mentioned gauge theories exists neatly only in the perturbative scheme. When and if the coupling constants involved cease to be small enough to be treated as an expansion parameter, theoretical predictions are increasingly hard to make. In some cases, meaningful predictions of this kind do not even exist. Dynamical chiral symmetry breaking (DCSB), confinement and the problem of bound states are some examples of the phenomena which are non perturbative in nature, foreign to an otherwise wide domain of perturbation theory. It is for this reason that a true understanding of these phenomena has so far illuded us. Among several other methods including lattice approach, Schwinger-Dyson equations (SDEs), [1, 2], provide a natural platform for continuum studies of non perturbative phenomena. Although the above-mentioned problems of this kind are of direct phenomenological relevance in the realm of QCD, [3, 4, 5, 6, 7] the corresponding SDEs are very complicated and it becomes relatively more difficult to gain an understanding of its theoretical intricacies. Within the scope of the present review, we concentrate only on the study of DCSB in QED and the problems associated with the loss of gauge invariance (GI).

We start with a brief introduction to the Schwinger-Dyson equations to set the scene and introduce the notation. In section 3, we recall dynamical generation of fermion masses under the simplest of approximations. These approximations result in the loss of gauge invariance. In section 4, we discuss in detail possible sources of this problem. In next sections, we take up the developments so far in the resolution of these issues. At the end, we present our conclusions.

2 Schwinger-Dyson Equations

These equations are an infinite set of coupled integral relations among the Green functions of the theory, and form equations of motion of the corresponding quantum field theory (QFT) [1, 2]. The two-point Green functions are related to the three-point functions, the three-point functions are related to the four-point functions and so on. In perturbation theory, fermion propagator can be expressed as in Figure 1.

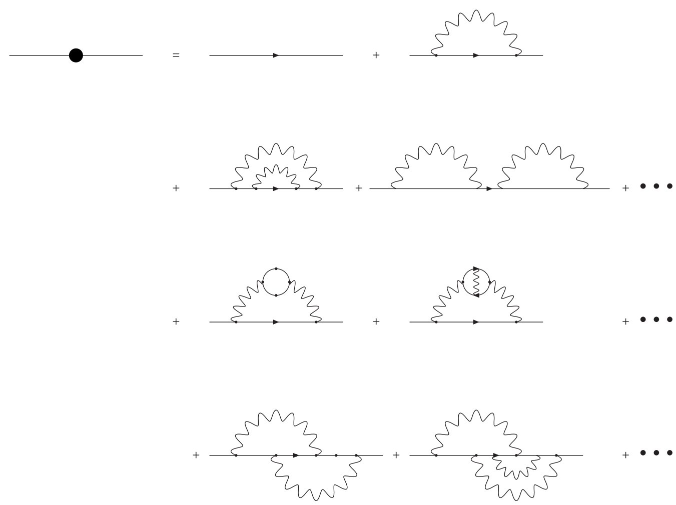
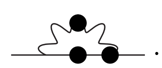


Figure 1. Perturbative expansion of the fermion propagator.

The left hand side corresponds to the full two-point function (the blob over the propagator indicates so), whilst on the right hand side, we find in the first place the bare propagator, which describes a fermion propagating with no perturbative corrections, followed by an infinite set of diagrams for three different types of perturbative corrections: to the fermion propagator itself, to the boson propagator and to the fermion-boson vertex. To be able to perform this infinite sum it is convenient to define $\Sigma(p)$ as :

$$\Sigma(p) \quad \equiv \tag{1}$$



This is called the self energy and involves all the corrections (as indicated by the blob) to the fermion propagator, to the boson propagator and to the fermion-boson vertex. In terms of self energy, perturbative expansion of the fermion propagator is despicted in Figure 2,

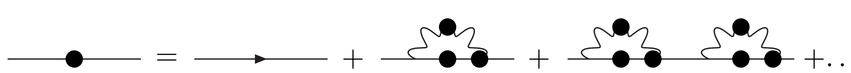


Figure 2. Perturbative expansion of the fermion propagator in terms of the self energy.

and corresponds to the sum

$$S_F(p) = S_F^0(p) + S_F^0(p)\Sigma(p)S_F^0(p) + S_F^0\Sigma(p)S_F^0(p)\Sigma(p)S_F^0(p) + \ldots\ . \tag{2}$$

Factorizing $S_F^0(p)$, the remaining geometric series yields

$$S_F(p) = \frac{S_F^0(p)}{1 - \Sigma(p) S_F^0(p)} \ . \tag{3}$$

Equivalently, leaving the first term apart and factorizing $S_F^0 \Sigma(p) S_F^0(p)$, the remaining terms form once again a geometric series, which after summation yields

$$S_F(p) = S_F^0(p) + S_F^0(p)\Sigma(p) \frac{S_F^0(p)}{1 - \Sigma(p) S_F^0(p)} \ . \tag{4}$$

Comparing (3) and (4) we obtain

$$S_F(p) = S_F^0(p) + S_F^0(p)\Sigma(p) S_F(p) \ . \tag{5}$$

This is the SDE for the fermion propagator. It is customary to write and study the SDE for the inverse fermion propagator instead, which is

$$S_F^{-1}(p) = S_F^{0-1}(p) - \Sigma(p) \ , \tag{6}$$

and corresponds to the diagram

Figure 3. The SDE for the (inverse) fermion propagator.

Of course, the full Green functions involved in the self energy obey their own SDE. The one for the boson propagator is shown in Figure 4

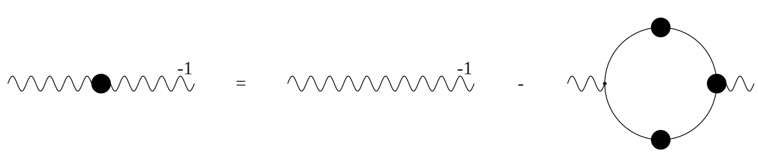


Figure 4. The SDE for the boson propagator.

and the one for the fermion-boson vertex is depicted in Figure 5.

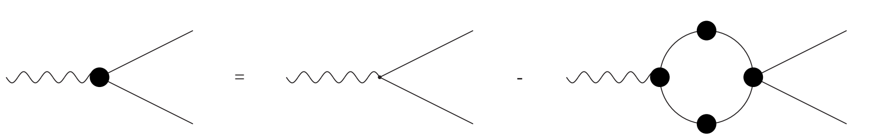


Figure 5. The SDE for the fermion-boson vertex.

We notice that the two-point functions are coupled to each other and both are coupled to a three-point function. The vertex, in turn, is coupled to two-point, three-point and four-point Green functions and so on, leading therefore, to an infinite set of coupled equations which form the field equations of the theory. Because of the Lorentz structure of Green functions the fermion propagator can be written in terms of two scalar functions. We prefer to write it as

$$S_F(p) = \frac{F(p)}{\not{p} - \mathcal{M}(p)} \ . \tag{7}$$

$F(p)$ is generally referred to as the wavefunction renormalization and $\mathcal{M}(p)$ as the mass function. Pole of the propagator corresponds to the physical mass of the fermion. The gauge boson propagator can in general be written as

$$\Delta^{\mu\nu}(q) = \frac{\mathcal{G}(q)}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \xi \frac{q^\mu q^\nu}{q^4} \ . \tag{8}$$

$\mathcal{G}(q)$ is called the gauge boson wavefunction renormalization and ξ is the usual covariant gauge parameter. The fermion-boson vertex $\Gamma^\mu(k,p)$ can be expressed in terms of twelve Lorentz structures as

$$\Gamma^\mu(k,p) = \sum_{i=1}^{12} v_i(k,p) V_i^\mu \ , \tag{9}$$

where

$$\begin{aligned}
&V_1^\mu = \gamma^\mu, && V_2^\mu = k^\mu, && V_3^\mu = p^\mu, \\
&V_4^\mu = \not{k}\gamma^\mu, && V_5^\mu = \not{k}k^\mu, && V_6^\mu = \not{k}p^\mu, \\
&V_7^\mu = \not{p}\gamma^\mu, && V_8^\mu = \not{p}k^\mu, && V_9^\mu = \not{p}p^\mu, \\
&V_{10}^\mu = \not{k}\not{p}\gamma^\mu, && V_{11}^\mu = \not{k}\not{p}k^\mu, && V_{12}^\mu = \not{k}\not{p}p^\mu.
\end{aligned} \tag{10}$$

From here, we can infer that the higher the order of the Green function, the more complicated is its structure and the corresponding SDE. In order to truncate the infinite tower of SDEs, special care should be taken to preserve gauge invariance and its implications.

If we want to truncate the infinite tower of SDEs at the level of two-point Green functions, a popular way is to make an *ansatz* for the fermion-boson interaction which preserves key features of QED, such as the gauge covariance of the Green functions and gauge invariance of associated physical observables. In the following, we shall review how some of the consequences of gauge invariance could put constraints on the non perturbative truncations of the SDE studies, namely the Ward-Green-Takahashi identities (WGTI) and the Landau-Khalatnikov-Fradkin transformations

(LKFT), and how perturbation theory, being the only known scheme where gauge invariance is satisfied at every order of approximation, could play a guiding role in this context.

3 Dynamical Chiral Symmetry Breaking

As we mentioned before, SDEs are an ideal framework to study non perturbative phenomena in gauge theories, such as dynamical breaking of chiral symmetry, which we shall study in QED. In order to proceed, we must recall that eq. (6) can be cast in arbitrary dimensions in the following way

$$S_F^{-1}(p) = S_F^{0\,-1}(p) - 4\pi i\alpha \int d^dk \gamma^\mu S_F(k) \Gamma^\nu(k,p) \Delta_{\mu\nu}(q) \; . \qquad (11)$$

To be able to solve it, we must know the explicit form of the gauge boson propagator and the fermion-boson vertex. However, we can simplify this equation in the so-called rainbow approximation by neglecting fermion loops (i.e., setting $\mathcal{G}(q) = 1$) and considering $\Gamma^\mu(k,p) = \gamma^\mu$. In such case, we are led to the expression

$$S_F^{-1}(p) = S_F^{0\,-1}(p) - 4\pi i\alpha \int d^dk \gamma^\mu S_F(k) \gamma^\nu \Delta^0_{\mu\nu}(q) \; . \qquad (12)$$

Diagramatically it corresponds to

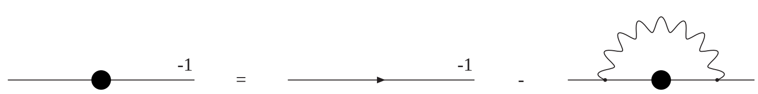


Figure 6. The SDE for the fermion propagator in rainbow approximation

and deserves its name because it correspond to a fermion propagator of the form

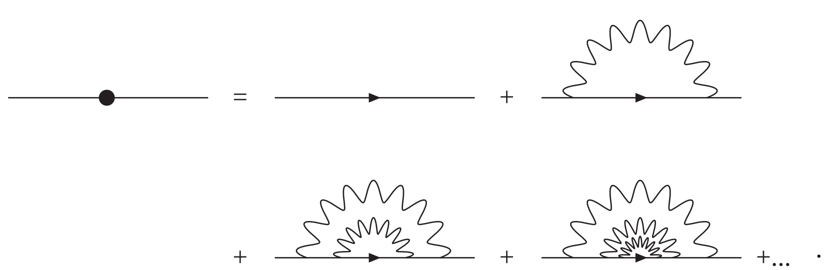


Figure 7. Fermion propagator in rainbow approximation.

This is probably the simplest approximation in which the DCSB can be studied. We shall recall the solution of eq. (12) in two cases: $d = 4$ and $d = 3$. In the former case, an extra complication occurs because of the presence of ultraviolet divergences. The numerical method is the simplest to employ if one uses the cut-off method to deal with these infinities. In this case, it could well be that not only the simplifying assumptions are the root of gauge invariance, but also the regularization procedure. The case of QED3 is neater in the sense that since it is a superrenormalizable theory, we do not have to deal with the possible gauge dependence problems arising from regularization issues. Instead, we can focus entirely on the assumptions made in the truncation of SDEs and how gauge symmetry and its implications can help us solve the problem of gauge invariance.

3.1 DCSB in QED4

Eq. (12) is a matrix equation which can be rearranged into a system of two coupled equations for $F(p)$ and $\mathcal{M}(p)$ on multiplying it by $\not{p}$ and 1, respectively, and taking the trace. After performing a Wick rotation to Euclidean space [1] and setting $d = 4$, angular integrations can be readily done. Radial integrations thus yield

$$\begin{aligned}
\frac{\mathcal{M}(p)}{F(p)} &= m_0 + \frac{\alpha}{4\pi}(3+\xi)\int_0^{\Lambda^2} dk^2 \frac{F(k)\mathcal{M}(k)}{k^2+\mathcal{M}^2(k)}\left[\frac{k^2}{p^2}\theta(p^2-k^2)+\theta(k^2-p^2)\right] \\
\frac{1}{F(p)} &= 1 + \frac{\alpha\xi}{4\pi}\int_0^{\Lambda^2} dk^2 \frac{F(k)}{k^2+\mathcal{M}^2(k)}\left[\frac{k^4}{p^4}\theta(p^2-k^2)+\theta(k^2-p^2)\right] , \qquad (13)
\end{aligned}$$

where Λ is the ultraviolet cut-off scale for the internal momentum and m_0 is the bare fermion mass. This system of equations has to be simultaneously solved by numerical techniques for every value of the gauge parameter in order to study the gauge dependence of the associated physical observables. One can study this system of coupled equations to look for dynamically generated mass, [9, 10, 11, 12]. We observe that in the Landau gauge ($\xi = 0$), $F(p) = 1$ which lead us to solve but one equation:

$$\mathcal{M}(p) = m_0 + \frac{3\alpha}{4\pi}\left[\frac{1}{p^2}\int_0^{p^2} dk^2 \frac{k^2\mathcal{M}(k)}{k^2+\mathcal{M}^2(k^2)} + \int_{p^2}^{\Lambda^2} \frac{\mathcal{M}(k^2)}{k^2+\mathcal{M}(k^2)}\right] . \qquad (14)$$

Setting $m_0 = 0$, results are shown in Figure 8. One can also translate eq. (14) into a differential equation with the appropriate boundary conditions. On doing so, we find

$$\frac{d}{dp^2}\left(p^4\frac{d\mathcal{M}(p)}{dp^2}\right) = -\frac{3\alpha}{4\pi}\frac{p^2\mathcal{M}(p)}{p^2+\mathcal{M}^2(p)} , \qquad (15)$$

with the boundary conditions

$$\frac{1}{p^2}\left[k^4\frac{d\mathcal{M}(k)}{dk^2}\right]_{k^2=0} \to 0 \qquad \text{and} \qquad \left[\mathcal{M}(k)+k^2\frac{d\mathcal{M}(k)}{dk^2}\right]_{k^2=\Lambda^2} \to 0 . \qquad (16)$$

[1] Attempts have been made to solve the SDEs in Minkowski space, see for example [8].

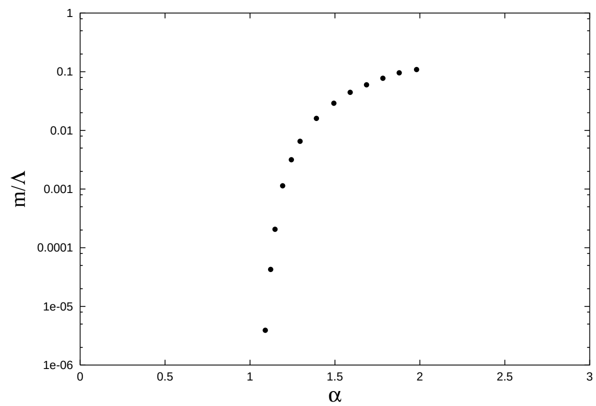


Figure 8. Dynamically generated mass as a function of the coupling.

In the limit $p^2 \gg \mathcal{M}^2(p)$, eq. (15) simplifies to

$$\frac{d}{dp^2}\left(p^4 \frac{d\mathcal{M}(p)}{dp^2}\right) \simeq -\frac{3\alpha}{4\pi}\mathcal{M}(p) \;, \tag{17}$$

which leads us to the following power-law behaviour for the mass function

$$\mathcal{M}(p) \sim (p^2)^s \qquad \text{with} \qquad s(s+1) = -\frac{3\alpha}{4\pi} \;. \tag{18}$$

Just like on the numerical graph, we observe that there is a critical value α_c of the coupling which distinguishes a pure power-law behaviour from an oscillatory behavior for the mass function:

$$\alpha_c = \frac{\pi}{3} \;. \tag{19}$$

This critical number tells us that in order for the fermion masses to be generated dynamically, we need a coupling constant above $\pi/3$. This critical coupling, in principle, is a physical observable. Unfortunately, as we shall see later, its value varies as a function of the gauge parameter which implies that our truncation procedure is not entirely reliable. We shall later come back to this problem and its possible solutions.

3.2 DCSB in QED3

Performing a similar exercise in eq. (12) for $d = 3$, we are left with the following system of equations

$$\begin{aligned} \frac{1}{F(p)} &= 1 - \frac{\alpha\xi}{4\pi}\int_0^\infty dk \frac{k^2 F(k)}{k^2+\mathcal{M}^2(k)}\left[1 - \frac{k^2+p^2}{2kp}\ln\left|\frac{k+p}{k-p}\right|\right] \;, \\ \frac{\mathcal{M}(p)}{F(p)} &= \frac{\alpha(\xi+2)}{\pi p}\int_0^\infty dk \frac{kF(k)\mathcal{M}(k)}{k^2+\mathcal{M}^2(k)}\ln\left|\frac{k+p}{k-p}\right| \;. \end{aligned} \tag{20}$$

Again, in Landau gauge we find $F(p) = 1$ and thus we are led to the equation

$$\mathcal{M}(p) = \frac{2\alpha}{\pi p} \int_0^\infty dk \frac{k\mathcal{M}(k)}{k^2 + \mathcal{M}^2(k)} \ln \left| \frac{k+p}{k-p} \right| . \tag{21}$$

Numerical solution of this equation also reveals dynamically generated mass function. However, unlike QED4, there is no critical value of coupling. If masses are generated for one value of coupling, they are also generated for any other value. Again, the large-momentum behaviour of the mass function can also be inferred by translating this equation into a differential equation with the appropriate boundary conditions. For this purpose, let us observe that we can make the approximation

$$\ln \left| \frac{k+p}{k-p} \right| \simeq \frac{2p}{k}\theta(k-p) + \frac{2k}{p}\theta(p-k) \tag{22}$$

which is valid for $k \gg p$ as well as for $k \ll p$. Thus, we can rewrite eq. (21) as

$$\frac{d}{dp}\left[p^3 \frac{d\mathcal{M}(p)}{dp}\right] + \frac{2}{\pi^2}\mathcal{M}(p) = 0 \tag{23}$$

with the boundary conditions

$$\left[p^3 \frac{d\mathcal{M}(p)}{dp}\right]_{p=0} \to 0 \qquad \text{and} \qquad \mathcal{M}(p)_{p\to\infty} \to 0 \, . \tag{24}$$

The solution to (23) is given by

$$\mathcal{M}(p) = \frac{4}{\pi^2 p}\left[c_1 J_2\left(\sqrt{\frac{8}{\pi^2 p}}\right) + c_2 Y_2\left(\sqrt{\frac{8}{\pi^2 p}}\right)\right] , \tag{25}$$

where $J(x)$ and $Y(x)$ are Bessel functions of the first and second kind, respectively. The second boundary condition imposes $c_2 = 0$. As $J(1/\sqrt{p}) \to 1/p$ for $p \to \infty$, we conclude that $\mathcal{M}(p) \to 1/p^2$ in this limit. c_1 cannot be determined from the equation and boundary conditions. The analysis for different values of the gauge parameter reveals an undesirable fact: the chiral condensate $\langle\bar{\psi}\psi\rangle$, which is a relevant physical observable in this case, depends also on the gauge parameter. The moral is that special care should be taken in the truncation of the tower of SDEs to preserve gauge invariance. The way to remedy this situation in the quenched case is to make use of an *ansatz* for the fermion-boson interaction which preserves key features of QED. These restrictions and their implementations are detailed in the next sections.

4 Restrictions on the Vertex

The constraints on the choice of the three-point vertex are as follows :

- **Ward Identities:**

  Ward and Ward-Green-Takahashi identities, i.e.,

$$\Gamma^\mu(p,p) = \frac{\partial}{\partial p_\mu} S_F^{-1}(p) \,, \tag{26}$$

$$q_\mu \Gamma^\mu(k,p) = S_F^{-1}(k) - S_F^{-1}(p) \,, \qquad q = k - p \tag{27}$$

  which relate the fermion propagator to the three-point vertex should be satisfied.

- **Gauge Independent Physical Observables:**

  Physical observables related with the fermion propagator such as the physical mass of the fermion, the condensate, etc. should be independent of the gauge parameter.

- **Gauge Independent Regularization Scheme:**

  Although it is not a constraint on the vertex and is only a technical detail, it is important enough to be taken carefully into consideration. A gauge dependent ultraviolet regulator used carelessly can add additional and incorrect gauge dependence in the study of Schwinger-Dyson equations.

- **Kinematic Singularities:**

  It should be independent of any kinematic singularities such as the ones when $k^2 \to p^2$.

- **Transformation Under C, P and T:**

  Under the operations of charge conjugation, parity and time reversal, it should transform in the same way as the bare vertex γ^μ.

- **Correct Weak Coupling Limit:**

  It should reduce to the perturbative Feynman expansion for the vertex in the limit when the coupling is small.

- **Landau-Khalatnikov-Fradkin Transformation Laws:**

  Under a variation of gauge, the fermion propagator and the fermion-boson vertex should change in accordance with the Landau-Khalatnikov-Fradkin transformations (LKFT). In the position space, the LKFT for the fermion propagator reads :

$$S(x;\xi) = S(x;0)\, \mathrm{e}^{-[\Delta_d(0)-\Delta_d(x)]} \,, \tag{28}$$

  where

$$\Delta_d(x) = \frac{\xi e^2}{16\pi^{d/2}} \, (\mu x)^{4-d} \, \Gamma\left(\frac{d}{2} - 2\right) \,.$$

The corresponding law for the vertex is more complicated in form.

5 The Ward Identities

As a consequence of the gauge invariance principle of QED, its Green functions are related through Ward identities, [13, 14, 15]. Particularly, the fermion propagator is related to the fermion-boson vertex, eq. (27). This identity is non-perturbative. Any choice of the vertex which does not satisfy this relation will violate the gauge principle. Let us first look at the bare vertex, i.e., $\Gamma^\mu(k,p) = \gamma^\mu$. In this case

$$\not{q} \;=\; \frac{\not{k}}{F(k)} - \frac{\not{p}}{F(p)} - \frac{\mathcal{M}(k)}{F(k)} + \frac{\mathcal{M}(p)}{F(p)} \;.$$

Obviously, this equation cannot be satisfied in every gauge for the dynamically generated fermion propagator. What about the Landau gauge? We know that in the Landau gauge $F(k) = F(p) = 1$. Moreover, the mass function is nearly flat below the pole position $\mathcal{M}^2(p) = p^2$. One can then conclude that the bare vertex practically satisfies the WGTI for the chirally asymmetric fermion propagator in the Landau gauge for the values of k^2 and p^2 below the physical (Euclidean) pole mass of the fermion. Therefore, if this is the physical observable we are interested to calculate, bare vertex in the Landau gauge should be sufficient as far as the WGTI is concerned. Will this approximation lead to the correct result? The answer is: not necessarily, although one may hope to be in its vicinity. The reason is the following. Imagine we could construct a vertex (which is easy to do in practice) which in fact satisfies the WGTI in every gauge exactly. Now we are free to do the calculation in any gauge. When we do this exercise, we find that the result for the physical observables in general varies from gauge to gauge. Therefore, having the satisfaction that the bare vertex satisfies the WGTI in the Landau gauge can not translate into a tangible statement for the gauge independence of physical observables. However, WGTI is a necessary condition and it must be satisfied by any vertex constructed.

The structure of the WGTI is such that we can decompose the full vertex into the following two components, longitudinal and transverse :

$$\Gamma^\mu(k,p) = \Gamma^\mu_L(k,p) + \Gamma^\mu_T(k,p) \;, \tag{29}$$

where, by definition the transverse vertex $\Gamma^\mu_T(k,p)$ is such that $q_\mu \Gamma^\mu_T = 0$. A possible choice for the longitudinal vertex is, [16, 17] :

$$\Gamma^\mu_L(k,p) = \frac{q^\mu}{q^2}[S_F^{-1}(k) - S_F^{-1}(p)] \;. \tag{30}$$

However, this has a singularity for $q^2 \to 0$. No such singularity is encountered in perturbation theory for the vertex at the one loop level, which makes it an unacceptable choice. As the longitudinal vertex is not unique, several proposals exist

in literature, [18, 19, 20]. More recently, it has been a common practice to employ as the longitudinal vertex the one proposed by Ball and Chiu, [18]. This *ansatz,* is based upon the Ward identity, eq. (26). Substituting in it the expression for the full fermion propagator, we observe that

$$\Gamma^\mu(p,p) = \frac{\gamma^\mu}{F(p)} + 2p^\mu \not{p} \frac{\partial}{\partial p^2} - 2p^\mu \frac{\partial}{\partial p^2} \frac{\mathcal{M}(p)}{F(p)} \ . \tag{31}$$

After we make the symmetric substitutions for the case when $k \neq p$

$$\begin{aligned}
\frac{1}{F(p)} &\to \frac{1}{2}\left[\frac{1}{F(k)} + \frac{1}{F(p)}\right] , \\
p^\mu &\to \frac{1}{2}(k^\mu + p^\mu) , \\
\not{p} &\to \frac{1}{2}(\not{k} + \not{p}) , \\
\frac{\partial}{\partial p^2}\frac{1}{F(p)} &\to \frac{1}{k^2 - p^2}\left[\frac{1}{F(k)} - \frac{1}{F(p)}\right] , \\
\frac{\partial}{\partial p^2}\frac{\mathcal{M}(p)}{F(p)} &\to \frac{1}{k^2 - p^2}\left[\frac{\mathcal{M}(k)}{F(k)} - \frac{\mathcal{M}(p)}{F(p)}\right] ,
\end{aligned} \tag{32}$$

Ball and Chiu defined the longitudinal vertex as, [18], :

$$\begin{aligned}
\Gamma^\mu_{BC} &= \frac{\gamma^\mu}{2}\left[\frac{1}{F(k)} + \frac{1}{F(p)}\right] + \frac{1}{2}\frac{(\not{k} + \not{p})(k+p)^\mu}{(k^2 - p^2)}\left[\frac{1}{F(k)} - \frac{1}{F(p)}\right] \\
&+ \frac{(k+p)^\mu}{(k^2 - p^2)}\left[\frac{\mathcal{M}(k)}{F(k)} - \frac{\mathcal{M}(p)}{F(p)}\right] .
\end{aligned} \tag{33}$$

This choice is free of any unwanted kinematic singularities. What about the transverse part of the vertex? Does it also have some simple connection with the fermion propagator or to expect and look for such a relation is a wild goose chase? It is as important a question as its answer unclear. Y. Takahashi, [21], discovered generalized Ward identitues which put additional constraints on the transverse vertex [22, 23, 24]. However, there is an alternative route based upon the arguments of multiplicative renormalizability in this connection. In a recent one loop perturbative calculation in arbitrary dimensions [25], generalizing the results earlier known in 3 and 4 dimensions, [26, 27, 28], it has been shown that in the massless case in the limit when $k^2 >> p^2$, [2]

$$\Gamma^\mu_T(k,p) = \frac{1}{2k^2}\left[\frac{S_F^{-1}(k)}{\not{k}} - \frac{S_F^{-1}(p)}{\not{p}}\right] \left[k^2\gamma^\mu - k^\mu \not{k}\right] \ . \tag{34}$$

[2]Note that a typing mistake was done in Eq. (26) of [25]. ξ should not be there in the denominator.

This is very much like the longitudinal vertex written in terms of the fermion propagator, guided by the WGTI. There is of course no guarantee that this relationship will survive higher order perturbative calculations. However, if such relations exist, they will be of enormous help in having a peep at the structure of the transverse vertex.

6 Physical Observables

We recall that the transverse part of the vertex plays a crucial rule to ensure the gauge independene of physical observables. Therefore, it is appropriate at this point to mention that without loss of generality, the transverse vertex can be expressed as [3]:

$$\Gamma_T^\mu(k,p) = \sum_{i=1}^{8} \tau_i(k^2,p^2,q^2) T_i^\mu(k,p) \ , \tag{35}$$

with an appropriate $\{T^\mu\}$ basis, defined first by Ball and Chiu, [18], as follows :

$$\begin{aligned}
T_1^\mu &= p^\mu(k\cdot q) - k^\mu(p\cdot q) \\
T_2^\mu &= \left[p^\mu(k\cdot q) - k^\mu(p\cdot q)\right](\not{k}+\not{p}) \\
T_3^\mu &= q^2\gamma^\mu - q^\mu \not{q} \\
T_4^\mu &= \left[p^\mu(k\cdot q) - k^\mu(p\cdot q)\right] k^\lambda p^\nu \sigma_{\lambda\nu} \\
T_5^\mu &= q_\nu \sigma^{\nu\mu} \\
T_6^\mu &= \gamma^\mu(p^2-k^2) + (p+k)^\mu \not{q} \\
T_7^\mu &= \frac{1}{2}(p^2-k^2)\left[\gamma^\mu(\not{p}+\not{k}) - p^\mu - k^\mu\right] + (k+p)^\mu k^\lambda p^\nu \sigma_{\lambda\nu} \\
T_8^\mu &= -\gamma^\mu k^\nu p^\lambda \sigma_{\nu\lambda} + k^\mu \not{p} - p^\mu \not{k}
\end{aligned} \tag{36}$$

$$\text{with} \qquad \sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \quad . \tag{37}$$

Gauge independence of physical observables should be the ultimate goal in non perturbative studies of SDE. If any proposal for the non perturbative form of the fermion-boson vertex is meant to be reliable, it would, certainly lead to gauge independent predictions for physical observables.

6.1 Proposals for the Vertex

Before going any further, it is important to present different proposals for the vertex found in literature both for QED4 and QED3.

[3]We shall come back to the discussion of this basis in the light of perturbation theory and the kinematic singularities.

- **Curtis and Pennington:**

  The vertex proposed by Curtis and Pennington, [28], in QED4 is

$$\Gamma^{\mu}_{CP} = \Gamma^{\mu}_{BC} + \Gamma^{\mu T}_{CP}$$

where

$$\Gamma^{\mu T}_{CP} = \tau_6(k^2,p^2) T^{\mu}_6(k,p,q) \tag{38}$$

with

$$\tau_6(k^2,p^2) = \frac{1}{2}\left[\frac{1}{F(k)} - \frac{1}{F(p)}\right] \frac{1}{d(k^2,p^2)}\,, \tag{39}$$

and

$$d(k^2,p^2) = \frac{(k^2-p^2)^2 + [\mathcal{M}^2(k)+\mathcal{M}^2(p)]^2}{k^2+p^2}\,.$$

In the massless case, it reduces to

$$\tau_6 = \frac{1}{2}\frac{k^2+p^2}{(k^2-p^2)^2}\left[\frac{1}{F(k)} - \frac{1}{F(p)}\right]\,. \tag{40}$$

- **Bashir and Pennington I:**

  The following vertex was proposed by Bashir and Pennington [29], also for QED4:

$$\Gamma^{\mu}_{BP_I} = \Gamma^{\mu}_{BC} + \Gamma^{\mu T}_{BP_I}$$

where

$$\Gamma^{\mu T}_{BP_I} = \sum_{i=2,3,6,8} \tau_i(k^2,p^2) T^{\mu}_i(k,p) \tag{41}$$

and, in the massless case,

$$\begin{aligned}
\tau_2(k^2,p^2) &= a_2 \frac{1}{(k^2+p^2)(k^2-p^2)}\left[\frac{1}{F(k)} - \frac{1}{F(p)}\right]\,,\\
\tau_3(k^2,p^2) &= a_3 \frac{1}{(k^2-p^2)}\left[\frac{1}{F(k)} - \frac{1}{F(p)}\right]\,,\\
\tau_6(k^2,p^2) &= a_6 \frac{(k^2+p^2)}{(k^2-p^2)^2}\left[\frac{1}{F(k)} - \frac{1}{F(p)}\right]\,,\\
\tau_8(k^2,p^2) &= a_8 \frac{1}{(k^2-p^2)}\left[\frac{1}{F(k)} - \frac{1}{F(p)}\right]\,,
\end{aligned}$$

with the constraints $1 + a_2 + 2a_3 + 2a_8 - 2a_6 = 0$ and $a_3 + a_6 = 1/2$, arising by demanding multiplicative renormalizability of the massless fermion propagator. Choice of Curtis and Pennington is $a_6 = 1/2, a_2 = 0$. The choice of Bashir and Pennington is $a_6 = -1/2$ and $a_2 = 2.75$.

- **<u>Bashir and Pennington II:</u>**

  A further generalization of the CP-vertex was carried out by Bashir and Pennington, [30, 31], where, in the expansion

$$\Gamma^{\mu T}_{BP_{II}} = \sum_{i=2,3,6,8} \tau_i(k^2, p^2) T_i^\mu(k, p) \; , \tag{42}$$

  a more complicated structure for the $\tau_i(k^2, p^2)$ was proposed in order to constrain the vertex to yield gauge independent answer for α_c in QED4.

- **<u>Dong, Munczek and Roberts:</u>**

  Dong, Munczek and Roberts, [32] made an attempt to propose alternative choices in arbitrary dimensions which would also achieve multiplicative renormalizability of the fermion propagator in 4 dimensions.

$$\Gamma^{\mu T}_{DMR} = \sum_{i=2,3,6,8} \tau_i(k^2, p^2) T_i^\mu(k, p) \; , \tag{43}$$

  where

$$\tau_i = \frac{1}{k^2 - p^2} \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right] f^i \; . \tag{44}$$

  The transverse coefficients f^i are defined as :

$$\begin{aligned} f^i &= 0 \, , \qquad\qquad \text{for} \quad i \neq 3, 8 \\ f^3 &= \frac{1}{2} \left(\frac{d}{2} - 1 \right) f^8 \, , \\ f^8 &= \frac{1}{\frac{d}{2} - 1} \frac{(d-1) I_3}{I_1 - I_3} \, , \end{aligned} \tag{45}$$

  where d is the number of space-time dimensions and

$$\begin{aligned} I_1 &= k^2 p^2 \mathcal{I}_1 \\ I_2 &= \frac{1}{2} \big((k^2 + p^2) \mathcal{I}_1 - 1 \big) \\ I_3 &= \frac{1}{2} [k^2 + p^2] I_2 \, , \end{aligned} \tag{46}$$

  with $\mathcal{I}_n = \int d\Omega_d 1/(k-p)^{2n}$.

- **<u>Bashir, Kızılersü and Pennington:</u>**

  Bashir, Kızılersü and Pennington [27] present the most general non perturbative construction of the transverse vertex required by the multiplicative renormalizability of the fermion propagator.

- **Burden and Roberts:**

  In the context of quenched QED3, Burden and Roberts have parametrized a slight modification to the BC-vertex in the following way, [33] :

$$
\begin{aligned}
\Gamma^{\mu}_{BR} &= \left[a\frac{1}{F(k)} + (1-a)\frac{1}{F(p)}\right]\gamma^{\mu} \\
&\quad + \frac{(k+p)^{\mu}((1-a)\not{k} - a\not{p})}{k^2-p^2}\left[\frac{1}{F(k)} - \frac{1}{F(p)}\right] \\
&\quad - \frac{(k+p)^{\mu}}{k^2-p^2}\left[\frac{\mathcal{M}(k)}{F(k)} - \frac{\mathcal{M}(p)}{F(p)}\right] \, , \qquad (47)
\end{aligned}
$$

  where a is a free parameter.

- **Burden and Tjiang:**

  Using a similar reasoning, Burden and Tjiang, [34], have deconstructed a one-parameter family of *ansätze* for massless QED3.

$$
\Gamma^{\mu T}_{BT} = \sum_{i=2,3,6,8} \tau_i(k^2,p^2) T^{\mu}_i(k,p) \, , \qquad (48)
$$

  Parametrizing the τ_i in exactly the same fashion as Dong, Munczek and Roberts, [32], and defining

$$
\bar{\tau} = \tau_8 + (k^2+p^2)\tau_2 \, , \qquad (49)
$$

  and $\bar{f}$ identically, they choose

$$
\begin{aligned}
\bar{f} &= -2(1+\beta)\frac{I(k,p)}{J(k,p)} \\
f^3 &= -\beta\frac{I(k,p)}{J(k,p)} \\
f^6 &= 0 \, , \qquad (50)
\end{aligned}
$$

  where

$$
\begin{aligned}
I(k,p) &= \frac{(k^2+p^2)^2}{16kp}\ln\left(\frac{(k+p)^2}{(k-p)^2}\right) - \frac{1}{4}(k^2+p^2) \\
J(k,p) &= \frac{(k^2-p^2)^2}{16kp}\ln\left(\frac{(k+p)^2}{(k-p)^2}\right) - \frac{1}{4}(k^2+p^2) \, . \qquad (51)
\end{aligned}
$$

  A choice of $\beta = 1$ leads to the DMR-vertex.

- **Bashir and Raya:**
More recently, an *ansatz* for the fermion-boson vertex in massive QED3 has been given by the authors [35]. Based on perturbation theory at one loop level, the transverse coefficients are found to have the form

$$\tau_i(k,p) \;=\; \alpha g_i \left[\sum_j^5 a_{ij}(k,p) I(l_j) + \frac{a_{i6}(k,p)}{k^2 p^2} \right] \;, \qquad i = 1, \cdots 8 \;, \tag{52}$$

where $l_1^2 = \eta_1^2 \chi/4$, $l_2^2 = \eta_2^2 \chi/4$, $l_3^2 = k^2$, $l_4^2 = p^2$ and $l_5^2 = q^2/4$. Functions η_1, η_2 and χ have been defined as

$$\begin{aligned} \chi &= m_0^2 (k^2 - p^2)^2 + q^2 (m_0^2 - k^2)(m_0^2 - p^2) \;, \\ \eta_1 &= -\left\{ \frac{m_0^2 (k^2 - p^2)(2m_0^2 - k^2 - p^2) + \chi}{2\chi (m_0^2 - k^2)} \right\} \;, \\ \eta_2 &= \left[\frac{\chi - m_0^2 (k^2 - p^2)(2m_0^2 - k^2 - p^2)}{\chi (m_0^2 - p^2)} \right] \;. \end{aligned} \tag{53}$$

The factors g_i are $-g_1 = m_0 \Delta^2 g_2 = 2m_0 \Delta^2 g_3 = 2\Delta^2 g_4 = g_5 = 2m_0 \Delta^2 g_6 = \Delta^2 g_7 = m_0 g_8 = m_0/4\Delta^2$, with $\Delta^2 = (k \cdot p)^2 - k^2 p^2$. The coefficients a_{ij} in the one loop perturbative expansion of the τ_i, eq. (52), are tabulated in [35]. This allow us to write a non perturbative form of τ_i as :

$$\begin{aligned} \tau_i = g_i \Bigg\{ & \sum_{j=1}^5 \left(\frac{2 a_{ij}(k,p) l_j^2}{\xi (m_0^2 + l_j^2)} \left[\frac{\xi}{2(\xi+2) l_j^2 I(l_j)} \left(\frac{\mathcal{M}(l_j)}{F(l_j)} - m_0 \right) - \left(1 - \frac{1}{F(l_j)} \right) \right] \right) \\ & + \frac{2 a_{i6}(k,p)}{\xi \left[k^2 A(p) - p^2 A(k) \right]} \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right] \Bigg\} \end{aligned} \tag{54}$$

with the notation

$$\begin{aligned} I(p) &= \frac{1}{\sqrt{-p^2}} \arctan \sqrt{\frac{-p^2}{m_0^2}} \;, \\ A(p) &= m_0 - (m_0^2 + p^2) I(p) \;. \end{aligned} \tag{55}$$

The transverse vertex is written solely as a function of the fermion propagator. Therefore, effectively, one has a WGTI for this part of the vertex. A computational difficulty to use the above vertex in SDE studies could arise as the unknown functions F and $\mathcal{M}$ depend on the angle between k and p. This problem can be circumvented by defining an effective vertex which shifts the angular dependence from the unknown functions F and $\mathcal{M}$ to the known basic functions of k and p. This is done by re-writing eq. (52), as follows :

$$\tau_i(k,p) \;=\; \alpha g_i \left[b_{i1}(k,p) I(k) + b_{i2}(k,p) I(p) + \frac{a_{i6}(k,p)}{k^2 p^2} \right] \;, \tag{56}$$

where

$$b_{i1}(k,p) = a_{i1}(k,p)\,\frac{I(l_1)}{I(l_3)} + a_{i3}(k,p) + \frac{1}{2}\,a_{i5}(k,p)\frac{I(l_5)}{I(l_3)}\,, \tag{57}$$

$$b_{i2}(k,p) = a_{i2}(k,p)\,\frac{I(l_2)}{I(l_4)} + a_{i4}(k,p) + \frac{1}{2}\,a_{i5}(k,p)\frac{I(l_5)}{I(l_4)}\,. \tag{58}$$

This form can be raised to a non-perturbative level exactly as before, with the only difference that the functions F and $\mathcal{M}$ are independent of the angle between the momenta k and p :

$$\tau_i = g_i\left\{\sum_{j=1}^{2}\left(\frac{2b_{ij}(k,p)\kappa_j^2}{\xi(m_0^2+\kappa_j^2)}\left[\frac{\xi}{2(\xi+2)\kappa_j^2 I(\kappa_j)}\left(\frac{\mathcal{M}(\kappa_j)}{F(\kappa_j)}-m_0\right)-\left(1-\frac{1}{F(\kappa_j)}\right)\right]\right) + \frac{2a_{i6}(k,p)}{\xi\left[k^2A(p)-p^2A(k)\right]}\left[\frac{1}{F(k)}-\frac{1}{F(p)}\right]\right\}, \tag{59}$$

where $\kappa_1^2 = k^2$ and $\kappa_2^2 = p^2$.

6.2 Physical Observables in QED4

In the previous section, we discussed several proposals for the 3-point vertex in QED. Below we see how they perform to achieve gauge independent physical observables associated with the DCSB.

- **The Bare Vertex:**

  The use of the bare vertex in various gauges results in rather different values of the critical coupling above which fermion mass is generated, Figure 9. This is physically unacceptable.

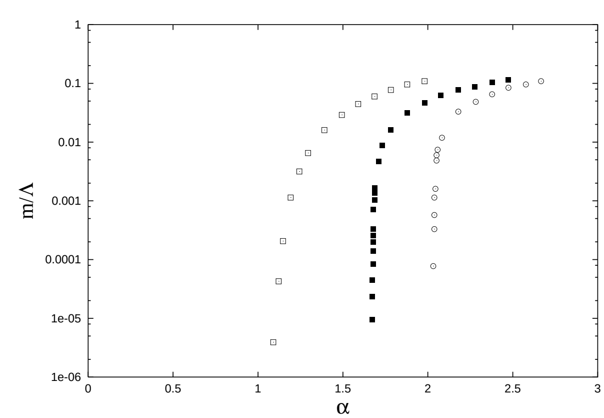


Figure 9. Critical coupling in various gauges employing the bare vertex.

- **Curtis and Pennington:**

  Using the bifurcation analysis, one is led to the conclusion that in dramatic contrast to the rainbow approximation, the critical coupling found with the CP-vertex [28] is only weakly gauge dependent in the neighbourhood of the Landau gauge. The gauge dependence is reduced by about 50%, [28], Figure 10.

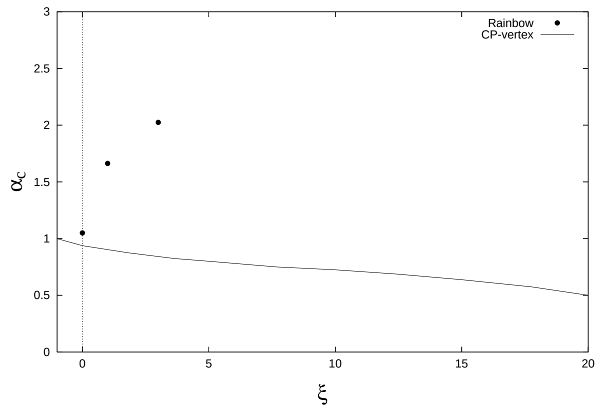


Figure 10. Critical coupling in various gauges employing the CP-vertex.

- **Bashir and Pennington I:**

  In Figure 11, we plot the gauge dependence of α_c by making use of the vertex proposed by Bashir and Pennington, [29]. For a comparison, the curve for the CP-vertex has also been plotted.

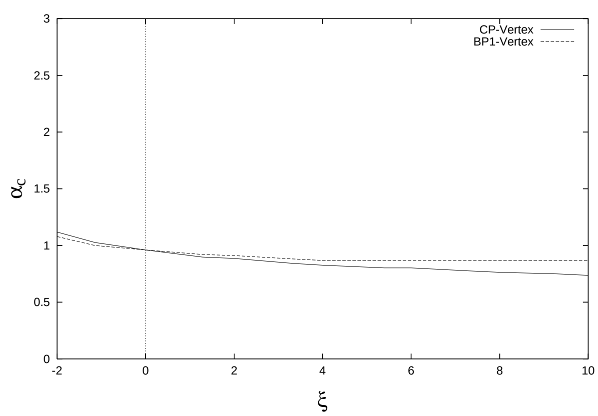


Figure 11. Critical coupling in various gauges employing the BP_I-vertex.

One notes that the improvement not only exists in the neighbourhood of the Landau gauge but that the curve continues to be flatter around $\alpha_c = 0.93$ even up to quite large values of the gauge parameter. Going from $\xi = 0$ to $\xi = 10$ reduces the gauge dependence by about 15%. The improvement becomes more significant when we are further away from the Landau gauge, Figure 12. For example, in going up to $\xi = 70$, the change in α_c is improved by more than 60% as compared to the CP-vertex. These results are encouraging in the sense that one finds a vertex which serves the aim of reducing the gauge dependence better than the ones constructed before, in particular the CP-vertex, without introducing any significant complication. But, however weak the variation of α_c with ξ may be, any gauge dependence shows that the BP I-vertex cannot be the exact choice. Even if it achieves a lot, it does not do it all.

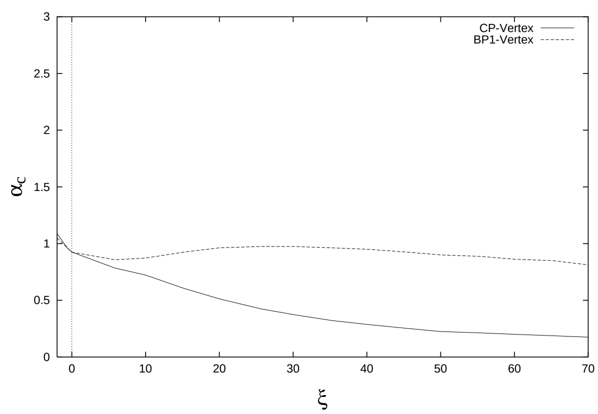


Figure 12. Critical coupling in various gauges employing the BP I-vertex.

- **Bashir and Pennington II:**

  The generalization of the vertex [29] developed in [30, 31] *renders the chiral symmetry breaking phase transition gauge independent in addition to ensuring that the massless fermion propagator is gauge covariant and multiplicatively renormalizable and also that the WGTI is satisfied.*

6.3 Physical Observables in QED3

In QED3, the coupling has dimensions of mass. As a result there is no critical value of coupling for the DCSB to take place. If it happens at one value of coupling, it takes place at every other value as well. The physical observables related to the DCSB are the the chiral condensate and the physical fermion mass. We review these quantities as predicted by the use of different vertex ansätze.

- **The Bare Vertex:**

  The gauge dependence of the chiral condensate using the bare vertex was studied by Burden and Roberts [33] in a narrow vicinity of the Landau gauge, namely from $\xi = 0$ to $\xi = 1$, where it seems to depend considerably on the choice of the gauge parameter. Later Bashir, Huet and Raya, [36], studied this dependence in a wider range of values of the gauge parameter, i.e., from $\xi = 0$ to $\xi = 5$. They find that with the increasing value of the gauge parameter, the gauge dependence gets reduced.

- **Burden and Roberts:**

  A slight modification to the Ball-Chiu vertex proposed by Burden and Roberts, [33], eq. (47), allows them to tune the parameter a to achieve a nearly gauge independent chiral condensate in the region $\xi = 0$ to $\xi = 1$. The value they report is $a = 0.53$ for this purpose.

- **Curtis and Pennington:**

  Strictly speaking, Curtis-Pennington vertex was discovered only for the 4-dimensional case. However, if one uses it to calculate the condensate in quenched QED3 in various gauges, [37], one gets the upper curve shown in Fig. 13.

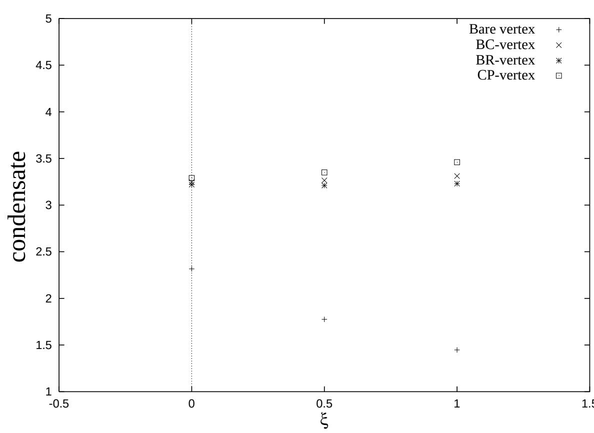


Figure 13. The condensate in quenched QED3 with various vertex ansätze.

7 SDEs and Dimensional Regularization

In the hunt for the non perturbative form of the fermion-boson vertex, the root of gauge invariance violation can be found in several places. We have earlier commented on the restrictions imposed on the vertex, but so far we have left an issue apart. In QED4, it is a common practice to use an ultraviolet cut-off to regulate the

divergent integrals. Such procedure, when employed carelessly, violates translational invariance (in momentum space) which leads to a violation of gauge invariance in the final results. This can be exemplified by calculating the one loop correction to the massless fermion propagator, which explicitly reads

$$\begin{aligned}\frac{1}{F(p)} &= 1+\frac{i\alpha}{4\pi^3 p^2}\int_M \frac{d^4k}{k^2q^4}\Bigg\{ -3k\cdot p(k^2+p^2)+4(k\cdot p)^2+2k^2p^2 \\ &\quad +\xi(k^2 p\cdot q - p^2 k\cdot q)\Bigg\}\,. \end{aligned} \tag{60}$$

Using an ultraviolet cut-off to regularize the integral, one finds

$$\frac{1}{F(p)} = 1-\frac{\alpha\xi}{4\pi}\ln\frac{p^2}{\Lambda^2}+\frac{\alpha\xi}{8\pi}\,. \tag{61}$$

The last term of this expression is spurious. It is known that such term can be removed by first making use of the WGTI in the SDE in the ξ-term and then carrying out the angular integration [38]. However, the fact remains that this method is ad hoc and it would be essential to confirm its validity. A reliable scheme of regularization of the divergent integrals is the dimensional regularization scheme. This approach is manifestly gauge invariant. These doubts were raised and addressed in [39, 40, 41]. However, the results obtained within dimensional regularization agree with those obtained by employing a correctly implemented cut-off within the numerical precision, and the numerical difficulties encountered in implementing this regulator has limited its further use, at least for the time being. It may be worth reminding at this point that QED3 lacks ultraviolet divergences. Therefore, it serves as an excellent laboratory to study the questions of gauge invariance without worrying about the procedure of regularization [36].

8 Kinematic Singularities

As mentioned before, the full vertex should be independent of unwanted kinematic singularities which are absent in perturbation theory. The WGTI is non singular as $k \to p$, and hence, the construction of the longitudinal piece of the vertex as proposed by Ball and Chiu ensures this condition is fulfilled by this part of the vertex. As a result, it also has no singularities when $k^2 \to p^2$. In the context of the longitudinal construction of vertices, the vertex proposed by Atkinson, Johnson, and Stam [16] exhibits kinematic singularities, whereas the one of Haeri [20] is free of them.

The transverse part of the vertex should also be independent of such singularities. It is natural to expect that a good choice for the basis to express the transverse part of the vertex is one that ensures that each coefficient is free of kinematic singularities. The original basis proposed by Ball and Chiu [18] was designed so that this

objective is achieved at the one loop order in the Feynman gauge ($\xi = 1$). Later a similar perturbative calculation in an arbitrary covariant gauge by Kızılersü, Reenders and Pennington [42] revealed that the coefficients of this basis develop kinematic singularities. They proposed a straightforward modification of the earlier basis that ensures each transverse component is separately free of kinematic singularities in any covariant gauge. The only modification takes place in the definition of T_4^μ which now reads :

$$T_4^\mu \quad = \quad q^2 \left[\gamma^\mu(\not{p}+\not{k}) - p^\mu - k^\mu\right] + 2(p-k)^\mu k^\lambda p^\nu \sigma_{\lambda\nu} \ . \tag{62}$$

This is also true in QED3, as it was demonstrated in Ref. [35], and in fact also in arbitrary dimensions which can be checked after a proper conversion of basis in the work of Davydychev *et. al.* [43].

- **Curtis and Pennington:**

  This widely implemented vertex *ansatz* does not exhibit kinematic singularities for massive fermions for which the vertex was proposed.

- **Burden and Roberts:**

  As a slight modification to the Ball-Chiu vertex, this *ansatz* is free of kinematic singularities.

- **Dong, Munczek, and Roberts:**

  This *ansatz* exhibits logarithmic singularities in 3 dimensions.

- **Burden and Tjiang:**

  This deconstruction of a family of vertex *ansätze* possesses logarithmic singularities.

- **Bashir and Pennington I & II:**

  This vertex proposal and its generalization do not exhibit kinematic singularities.

- **Bashir and Raya:**

  This family of vertex *ansätze* was constructed free of kinematical singularities.

9 Transformation Under C, P and T

Another constraint on the non perturbative form of the fermion-boson vertex arises from the requirement that it should transform in the same way the bare vertex does under C, P, and T. Thus, e.g., under the charge conjugation operation, we must have :

$$C\Gamma_\mu C^{-1} \quad = \quad -\Gamma_\mu^{\mathrm{T}}(-p,-k) \ ,$$

Now making use of the identities

$$C = -C^{\rm T} \qquad\qquad C\gamma_\mu C^{-1} = -\gamma_\mu^{\rm T} ,$$

it is easy to see that for the Ball and Chiu selection of the transverse basis vectors,

$$\begin{aligned} CT_i^\mu(k,p)C^{-1} &= -T_i^{\mu{\rm T}}(-p,-k) \qquad \text{for} \quad i\neq 6 \\ CT_6^\mu(k,p)C^{-1} &= T_6^{\mu{\rm T}}(-p,-k) \, . \end{aligned} \tag{63}$$

This implies that all the $\tau_i(k^2,p^2)$ are symmetric, except for $\tau_6(k^2,p^2)$ which is antisymmetric under $k^2 \leftrightarrow p^2$. With the choice of the transverse basis as proposed by Kızılersü, Reenders and Pennington [42], correct charge conjugation properties demands all of the τ_i to be symmetric under $k \leftrightarrow p$, except for τ_4 and τ_6 which should be antisymmetric. Most of the vertices proposed so far fulfil this requirement by construction, [16, 18, 20, 28, 30, 31, 32, 34], except the one presented by Burden and Roberts, [33], eq. (47). In this equation if we substitute $a = 1/2+\delta$, we observe that the BR-vertex can also be written as :

$$\begin{aligned} \Gamma^\mu_{BR} &= \Gamma^\mu_{BC} + \delta\left[\frac{1}{F(k)} - \frac{1}{F(p)}\right][k^2-p^2]\left\{\gamma^\mu(p^2-k^2) + (k+p)^\mu \not{q}\right\} \\ &= \Gamma^\mu_{BC} + \delta\tau_6 T_6^\mu \, , \end{aligned} \tag{64}$$

with

$$\tau_6 = (k^2-p^2)\left[\frac{1}{F(k)} - \frac{1}{F(p)}\right] \, . \tag{65}$$

As already mentioned, charge conjugation symmetry for the vertex requires τ_6 to be antisymmetric under the interchange of k and p, but the BR-vertex forces it to be symmetric.

10 Perturbation Theory

Interactions determine the dynamical structure of any theory. In the case of QED, interactions correspond to the fermion-boson vertex. The SDE for this vertex and the higher-order Green functions are very complicated and a non perturbative solution of these equations is a formidable task. As we have explicitly seen in previous sections, it is customary to make an *ansatz* for the vertex to solve the SDE for the fermion propagator. Therefore, an intelligent guess of its non perturbative form is a key issue. On the other hand, we know that perturbation theory (PT) is the only fully reliable scheme of truncation of the SDEs where all the key features of a gauge field theory, such as the gauge identities, are satisfied automatically at every order of approximation. Therefore, to stand a chance of maintaining these features at the non perturbative level, a connection with PT could play a vital role. Besides, for physically meaningful solutions of these equations, we must demand the results

from SDEs to agree with perturbative ones in the weak coupling regime. Therefore, PT can guide us in the search for the possible non perturbative forms of the fermion-boson interaction.

For the study of DCSB, the captivatingly simple use of the rainbow approximation is based on the lowest order perturbative expansion of the vertex. However, not surprisingly, it violates gauge invariance and its implications at the non perturbative level. Thus, for reliable results, we must look for more sophisticated *anzätze* which do respect these key features, peeping at the higher orders in perturbation theory. Recall that the full fermion-boson vertex can be expressed in terms of the twelve amplitudes given in eq. (10), which would suggest that it involves the same number of independent functions. However the non perturbative form of these functions should be related to the fermion propagator in the way dictated by the WGTI. Of course this identity constrains only the so called longitudinal part of the vertex, and hence only some of the twelve functions which define the full three-point function. Additional constraints can be set by demanding these functions to fulfil their corresponding LKFT, making sure that the dynamically generated fermion propagator does so as well. As in PT this is achieved automatically, one cannot resist the temptation to be guided by it into the non perturbative regime. The procedure is as follows : the longitudinal part of the vertex can be constrained by the perturbative expansion of the fermion propagator by means of the WGTI in terms of the Ball-Chiu vertex *ansatz*, eq. (33). On the other hand, the perturbative calculation of the fermion-boson vertex can allow us to identify the transverse part of the vertex by subtracting from it the previously found longitudinal piece. As pointed out in [42], this procedure should be performed in an arbitrary gauge, otherwise we will be clueless of the non perturbative form of the vertex. For instance, if we calculated the vertex in massless QED in the Landau gauge, we would find that the $v_1(k,p)$ function was like its bare form, just γ^μ, and we would have no information whatsoever of its non perturbative $\frac{1}{2}[1/F(k)+1/F(p)]\gamma^\mu$ structure. In the spirit of this reasoning, the analysis at one loop level was already performed in literature. Based on PT, we can compute the fermion propagator at the one loop level from the diagram

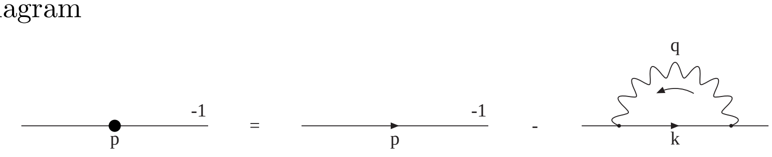


Figure 14 : One loop correction to the fermion propagator.

which yields the following expressions for $F(p)$ and $\mathcal{M}(p)$

$$
\begin{aligned}
\frac{1}{F(p)} &= \frac{e^2}{i(2\pi)^d}\frac{(d-2)\xi}{2p^2}\left[(p^2+m^2)\int\frac{d^dw}{(k-w)^2(w^2-m^2)}-\int\frac{d^dw}{w^2-m^2}\right],\\
\frac{\mathcal{M}(p)}{F(p)} &= \frac{e^2}{i(2\pi)^d}m(d-1-\xi)\int\frac{d^dw}{(k-w)^2(w^2-m^2)}\;. \qquad (66)
\end{aligned}
$$

Then, from the expression (33), we can set perturbative constraints to the longitudinal part of the vertex, thus achieving the first step of the analysis. The second step demands calculating the one loop correction to the fermion-boson vertex, which can be done from the diagram

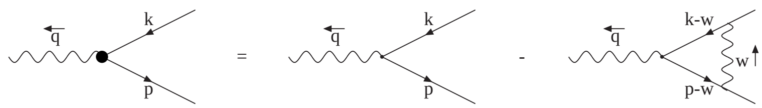


Figure 15 : One loop correction to the vertex.

and can be expressed as :

$$\Gamma^\mu(k,p) = \gamma^\mu + \Lambda^\mu \ . \tag{67}$$

The one loop correction can be written as

$$\begin{aligned} \Lambda^\mu &= -4i\pi\alpha \int \frac{d^dw}{(2\pi)^d} \left\{ \frac{A^\mu}{[(p-w)^2-m^2][(k-w)^2-m^2]w^2} \right. \\ &\quad \left. + \frac{B^\mu}{[(p-w)^2-m^2][(k-w)^2-m^2]w^4} \right\} , \end{aligned} \tag{68}$$

where

$$\begin{aligned} A^\mu &= \gamma^\alpha(\not{p}-\not{w})\gamma^\mu(\not{p}-\not{w})\gamma_\alpha \\ &\quad + m\gamma^\alpha[(\not{p}-\not{w})\gamma^\mu + \gamma^\mu(\not{k}-\not{w})]\gamma_\alpha + m^2\gamma^\alpha\gamma^\mu\gamma_\alpha \\ B^\mu &= \not{w}(\not{p}-\not{w})\gamma^\mu(\not{k}-\not{w})\not{w} \\ &\quad + m\not{w}[(\not{p}-\not{w})\gamma^\mu + \gamma^\mu(\not{k}-\not{w})]\not{w} + m^2\not{w}\gamma^\mu\not{w} \ . \end{aligned} \tag{69}$$

The master integrals to be calculated in this connection are :

$$\begin{aligned} K^{(0)} &= \int_M d^dw \, \frac{1}{[(p-w)^2-m^2]\,[(k-w)^2-m^2]} \\ J^{(0)} &= \int_M d^dw \, \frac{1}{w^2\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]} \\ J^{(1)}_\mu &= \int_M d^dw \, \frac{w_\mu}{w^2\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]} \\ J^{(2)}_{\mu\nu} &= \int_M d^dw \, \frac{w_\mu w_\nu}{w^2\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]} \\ I^{(0)} &= \int_M d^dw \, \frac{1}{w^4\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]} \\ I^{(1)}_\mu &= \int_M d^dw \, \frac{w_\mu}{w^4\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]} \\ I^{(2)}_{\mu\nu} &= \int_M d^dw \, \frac{w_\mu w_\nu}{w^4\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]} \ . \end{aligned} \tag{70}$$

There exist several works which carry out this calculation employing a varying degree of complication :

- Ball and Chiu, [18], evaluate the transverse vertex at the one loop level in the Feynman gauge for massive fermions (QED4).
- Curtis and Pennington, [28], evaluate the transverse vertex at the one loop level in an arbitrary covariant gauge in the limit when momentum in one of the fermion legs is much greater than in the other leg for massless fermions (QED4).
- Kızılersü, Pennington and Reenders, [42], evaluate the transverse vertex at the one loop level in an arbitrary covariant gauge for all regions of momenta for massive fermions (QED4).
- Bashir, Pennington and Kızılersü, [27], evaluate the transverse vertex at the one loop level in an arbitrary covariant gauge for all regions of momenta for massless fermions (QED3).
- Bashir and Raya, [35], evaluate the transverse vertex at the one loop level in an arbitrary covariant gauge for all regions of momenta for massive fermions (QED3).
- Davydychev, Osland and Sakset, [43], evaluate the transverse vertex at the one loop level in an arbitrary covariant gauge for all regions of momenta in massive QED in arbitrary dimensions.
- **Case** $d = 4$

  In the four dimensional case, the one loop perturbative correction to the fermion propagator was calculated in [42]. The authors find

$$
\begin{aligned}
F(p) &= 1 - \frac{\alpha\xi}{4\pi}\left[C\mu^{\epsilon} + \left(1 - \frac{m^2}{p^2}\right)(1-L)\right] , && (71)\\
\mathcal{M}(p) &= m + \frac{\alpha m}{\pi}\left[\left(1+\frac{\xi}{4}\right) + \frac{3}{4}(C\mu^{\epsilon} - L) + \frac{\xi}{4}\frac{m^2}{p^2}(1-L)\right] , && (72)
\end{aligned}
$$

  where

$$
\begin{aligned}
L &= \left(1+\frac{m^2}{p^2}\right)\ln\left(1+\frac{p^2}{m^2}\right) ,\\
C &= -\frac{2}{\epsilon} - \gamma - \ln\pi - \ln\left(\frac{m^2}{\mu^2}\right) . && (73)
\end{aligned}
$$

These results, in combination with eq. (33) yield the following perturbative expression for the one loop longitudinal vertex

$$\begin{aligned}\Gamma_L^\mu(k,p) &= \frac{\alpha\xi}{8\pi}\gamma^\mu\left[2C\mu^\epsilon + \left(1+\frac{m^2}{k^2}\right)(1-L') + \left(1+\frac{m^2}{p^2}\right)(1-L)\right] \\ &+\frac{\alpha\xi}{8\pi}\frac{(k+p)^\mu(\not k+\not p)}{(k^2-p^2)}\left[m^2\left(\frac{1}{k^2}-\frac{1}{p^2}\right) - \left(1+\frac{m^2}{k^2}\right)L' + \left(1+\frac{m^2}{p^2}\right)L\right] \\ &+\frac{\alpha m}{4\pi}(3+\xi)\frac{(p+k)^\mu}{(k^2-p^2)}[L-L'] \,, \qquad (74)\end{aligned}$$

where $L' = L(k \leftrightarrow p)$. On the other hand, the one loop correction to the vertex is found to be written as combination of the functions defined in eq. (73), of the function

$$S = \frac{1}{2}\sqrt{1-4\frac{m^2}{q^2}}\ln\left|\frac{\sqrt{1-4m^2/q^2}+1}{\sqrt{1-4m^2/q^2}-1}\right| \,, \qquad (75)$$

and J_0 which is a combination of Spence functions. We refer the reader to the original reference, [42] for the explicit lengthy expressions. A subtraction of eq. (74) from this correction gives the following expression for the transverse vertex

$$\begin{aligned}\Gamma_T^\mu(k,p) &= \frac{\alpha}{4\pi}\left\{\sum_{i=1}^{12} V_i^\mu\left(\frac{1}{2\Delta^2}[a_1^{(i)}-(\xi-1)a_2^{(i)}]J_A\right.\right. \\ &+\frac{1}{2\Delta^2}[b_1^{(i)}-(\xi-1)b_2^{(i)}]J_B \\ &+\frac{1}{2\Delta^2}[c_1^{(i)}-(\xi-1)c_2^{(i)}]I_A \\ &+\frac{1}{2\Delta^2}[d_1^{(i)}-(\xi-1)d_2^{(i)}]I_B \\ &+\frac{1}{2p^2(k^2-p^2)(p^2-m^2)\Delta^2}[e_1^{(i)}-(\xi-1)e_2^{(i)}]L \\ &+\frac{1}{2k^2(k^2-p^2)(k^2-m^2)\Delta^2}[f_1^{(i)}-(\xi-1)f_2^{(i)}]L' \\ &+\frac{1}{\Delta^2}[g_1^{(i)}-(\xi-1)g_2^{(i)}]S \\ &+\frac{1}{2\Delta^2}[h_1^{(i)}-(\xi-1)h_2^{(i)}]J_0 \\ &\left.\left.+\frac{1}{\Delta^2}[l_1^{(i)}-(\xi-1)l_2^{(i)}]\right)\right\} \,, \qquad (76)\end{aligned}$$

where $\Delta = (k\cdot p)^2 - k^2p^2$. The expressions J_A, J_B, I_A and I_B are combinations of L, L', S and J_0, and along with the coefficients $a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i$ and l_i are tabulated in Ref. [42]. It yields the results obtained by [18] and [28] as special cases.

- <u>**Case $d = 3$**</u>

  For QED3, the same exercise was done in [35] for the massive fermions. The authors find

$$\begin{aligned} \frac{1}{F(p)} &= 1 - \frac{\alpha\xi}{2p^2}\left[m - (m^2+p^2)\,I(p)\right] \,, \\ \frac{\mathcal{M}(p)}{F(p)} &= m\left[1 + \alpha(\xi+2)\,I(p)\right] \,, \end{aligned} \tag{77}$$

  with $I(p)$ defined as in eq. (55). With the aid of these expressions, the one loop longitudinal part of the vertex can be written as

$$\begin{aligned} \Gamma_L^\mu &= \left[1 + \frac{\alpha\xi}{4}\sigma_1\right]\gamma^\mu + \frac{\alpha\xi}{4}\sigma_2\left[k^\mu \not{k} + p^\mu \not{p} + k^\mu \not{p} + p^\mu \not{k}\right] \\ &+ \alpha(\xi+2)\sigma_3\left[k^\mu + p^\mu\right] \,, \end{aligned} \tag{78}$$

  where

$$\begin{aligned} \sigma_1 &= \frac{m^2+k^2}{k^2}\,I(k) + \frac{m^2+p^2}{p^2}\,I(p) - m\frac{k^2+p^2}{k^2p^2} \,, \\ \sigma_2 &= \frac{1}{k^2-p^2}\left[\frac{m^2+k^2}{k^2}\,I(k) - \frac{m^2+p^2}{p^2}\,I(p) + m\frac{k^2-p^2}{k^2p^2}\right] \,, \\ \sigma_3 &= m\,\left[I(k) - I(p)\right] \,. \end{aligned} \tag{79}$$

  The one loop correction for the vertex in this case is more simple, since it can be written in terms of functions $I(p)$, with complicated arguments though. For the identification of the transverse part of the vertex, the authors made use of the decomposition (35) for the vertex, making use of the basis vectors proposed by Kızılersü *et. al.* After a lengthy but straightforward algebra, the coefficients τ_i are identified, eq. (52). The main advantage of QED3 is that no special functions are involved and raising the perturbative expression to a no perturbative status is possible.

- <u>**Case of arbitrary dimensions**</u>

  Davydychev *et. al.*, [43], have calculated the one loop correction to the fermion propagator and to the fermion-boson vertex. These authors introduce two Master Integrals :

$$I(\nu_1,\nu_2,\nu_3) \equiv \int \frac{d^dw}{[(p-w)^2-m^2]^{\nu_1}[(k-w)^2-m^2]^{\nu_2}[w^2]^{\nu_3}}, \tag{80}$$

$$\mathcal{I}(\nu_1,\nu_2,\nu_3) \equiv \int \frac{d^dw}{[(p-w)^2]^{\nu_1}[(k-w)^2]^{\nu_2}[w^2-m^2]^{\nu_3}} \,. \tag{81}$$

In terms of these integrals, one can write out the fermion propagator and fermion-boson vertex in the context of QED in arbitrary dimensions. They find that the fermion propagator at one loop order can be written as :

$$
\begin{aligned}
\frac{1}{F(p)} &= \frac{e^2}{i(2\pi)^d}\frac{(d-2)\xi}{2p^2}\left[(p^2+m^2)\mathcal{I}(0,1,1)-\mathcal{I}(0,0,1)\right] , \\
\frac{\mathcal{M}(p)}{F(p)} &= \frac{e^2}{i(2\pi)^d} m(d-1-\xi)\mathcal{I}(0,1,1) \, . \qquad (82)
\end{aligned}
$$

The longitudinal part of the vertex can be straightforwardly obtained from eq. (33). The transverse coefficient are found to be

$$
\begin{aligned}
\tau_i &= \frac{e^2}{i(2\pi)^d}\Bigg\{ t_{i,0} I(1,1,1) + t_{i,1}[-(k\cdot q)\mathcal{I}(0,1,1)+(p\cdot q)\mathcal{I}(1,0,1) \\
&+ q^2 I(1,1,0)] + t_{i,2}(\mathcal{I}(0,1,1)+\mathcal{I}(1,0,1)-2I(1,1,0)) \\
&+ t_{i,3}\left(\mathcal{I}(0,1,1)+\mathcal{I}(1,0,1)-2\frac{\mathcal{I}(0,0,1)}{m^2}\right) \\
&+ t_{i,4}(\mathcal{I}(0,1,1)+\mathcal{I}(1,0,1)) + t_{i,5}\frac{\mathcal{I}(0,1,1)-\mathcal{I}(1,0,1)}{k^2-p^2}\Bigg\} , \qquad (83)
\end{aligned}
$$

where the $t_{i,j}$, $j = 1 \ldots 5$ are basic functions of k, p and q, listed in [43].

Because of the complexities of expressions, the non perturbative structure of the vertex can hardly be obtained from them in arbitrary dimensions. However, in 3 dimensions, the expressions simplify to an extent that a less complicated non perturbative structure of the transverse part of the fermion-boson interaction can be seen. In view of the perturbative requirements, we analyse below some of the *ansätze* proposed :

- **Curtis and Pennington :**

  The *ansatz* of Curtis and Pennington, [28] was explicitly constructed to reproduce the corresponding result in the weak coupling regime in the relevant kinematical domain when momentum in one fermion leg is much greater than in the other leg, i.e.,

$$
\Gamma_T^\mu(k,p) \simeq \frac{\alpha\xi}{8}\ln\left(\frac{k^2}{p^2}\right)\left[-\gamma^\mu + \frac{k^\mu \not{k}}{k^2}\right] \, . \qquad (84)
$$

- **Dong, Munczek, and Roberts :**

  This *ansatz* reproduces eq. (84) for 4 dimensions although it employs a different set of transverse coefficients as compared to the CP vertex.

- **Bashir and Pennington I :**

  It also satisfies eq. (84) in addition to reducing the gauge dependence of the critical coupling in comparison with the CP-vertex.

- **Bashir and Pennington II :**

  With the appropriate choice of the the constrained functions W_1 and W_2, defining the transverse vertex, it should match the perturbative results in the weak coupling regime.

- **Bashir, Kızılersü, and Pennington :**

  For the massless case, the constraint obtained from it would reproduce the wavefunction renormalization to all orders with the correct exponent.

For the case of QED3 specifically,

- **Curtis and Pennington :**

  The CP vertex which was designed for QED4 also reproduces correct perturbative vertex at the one loop in QED3 in the limit when momentum in one of the fermion legs is much greater than in the other :

$$\Gamma_T^\mu(k,p) \simeq \frac{\alpha\xi}{8}\frac{\pi}{\sqrt{p^2}}\left[-\gamma^\mu + \frac{k^\mu \not{k}}{k^2}\right] . \tag{85}$$

- **Burden and Roberts :**

  The choice of the parameter a which leads to an almost gauge independent chiral condensate from $\xi = 0 - 1$ has the undesirable feature of mismatch against perturbative results in the weak coupling regime even for the limit in eq. (85).

- **Burden and Tjiang :**

  The one parameter (β) family of vertex *ansätze* deconstructed by Burden and Tjiang was done assuming β to be gauge independent. The perturbative result eq. (85) shows that this could not be the case.

- **Bashir and Raya :**

  This *ansatz* was constructed to reproduce perturbative results in the weak coupling regime not only in the above-mentioned kinematical regime, but for all momentum regimes.

In the construction of non perturbative vertices, a next step could be to have a vertex in arbitrary dimensions reproducing the one loop vertex correctly. Going beyond one loop should come after that. Several advances have been made in solving the integrals which appear in the two loop calculation of the fermion propagator, [44], and the vertex, [45, 46, 47, 48, 49].

11 LKF Transformation Laws

In a gauge field theory, Green functions transform in a specific manner under a variation of gauge. In QED these transformations carry the name Landau-Khalatnikov-Fradkin transformations (LKFT), [50, 51, 52]. These were derived also by Johnson and Zumino through functional methods, [53, 54], and by others [55, 56, 57] [4]. LKFT are nonperturbative in nature and hence have the potential of playing an important role in addressing the problems of gauge invariance which plague the strong coupling studies of SDEs. In general, the rules governing these transformations are far from simple. The fact that they are written in coordinate space adds to their complexity. As a result, these transformations have played less significant and practical role in the study of SDE than desired.

As compared to the LKFT, WGTI are simpler to use and, therefore, have been extensively implemented in the SDE studies which are based either upon gauge technique, e.g., [59, 60, 61, 62, 63, 64, 65, 66, 67], or upon making an *ansatz* for the full fermion-boson vertex, e.g., [18, 28, 30, 31, 38, 68, 69, 70]. WGTI follow from the Becchi-Rouet-Stora-Tyutin (BRST) symmetry. One can enlarge these transformations by transforming also the gauge parameter ξ, [71, 72], to arrive at modified Ward identities, known as Nielsen identities (NI). An advantage of the NI over the conventional Ward identities is that $\partial/\partial\xi$ becomes a part of the new relations involving Green functions. This fact was exploited in [73, 74] to prove the gauge independence of some of the quantities related to two-point Green functions at the one loop level and to all orders in perturbation theory, respectively. As it is a difficult task to establish the gauge independence of physical observables in the study of SDE, NI may play a significant role in addressing this issue in addition to Ward identities and LKFT. However, in this section, we concentrate only on the LKFT.

The LKFT for the three-point vertex is complicated and hampers direct extraction of analytical restrictions on its structure. Burden and Roberts, [75], carried out a numerical analysis to compare the self-consistency of various *ansätze* for the vertex, [18, 28, 76], by means of its LKFT. In addition to these numerical constraints, indirect analytical insight can be obtained on the nonperturbative structure of the vertex by demanding correct gauge covariance properties of the fermion propagator. In the context of gauge technique, examples are [77, 78, 79]. Concerning the works based upon choosing a vertex *ansätze*, references [27, 28, 30, 31, 32, 34] employ this idea [5]. However, all the work in the later category has been carried out mostly for massless QED3 and QED4. The masslessness of the fermions implies that the fermion propagator can be written only in terms of one function, the so-called wavefunction renormalization, $F(p)$. In order to apply the LKFT, one needs

[4] Fukuda, Kubo and Yokoyama have looked for a possible formalism where renormalization constants of the wave function are in fact gauge invariant [58].

[5] A criticism of the vertex construction in [34] was raised in [26].

to know a Green function at least in one particular gauge. This is a formidable task. However, one can rely on approximations based on perturbation theory. It is customary to take $F(p) = 1$ in the Landau gauge, an approximation justified by one loop calculation of the massless fermion propagator in arbitrary dimensions, see for example, [43]. The LKFT then implies a power law for $F(p)$ in QED4 and a simple trigonometric function in QED3. To improve upon these results, one can take two paths

- incorporate the information contained in higher orders of perturbation theory both in the massless and massive case.
- study how the dynamically generated fermion propagator transforms under the LKFT.

As pointed out in [27], in QED4, the power law structure of the wavefunction renormalization remains intact by increasing order of approximation in perturbation theory although the exponent of course gets contribution from next to leading logarithms and so on [6]. In [27], constraint was obtained on the 3-point vertex by considering a power law where the exponent of this power law was not restricted only to the one loop fermion propagator. In QED3, the two loop fermion propagator was evaluated in [26, 81, 82], where it was explicitly shown that the the approximation $F(p) = 1$ is only valid up to one loop, thus violating the *transversality condition* advocated in [34]. The result found there was used used in [83] to find the improved LKF transform. In the following, we try to summarize in relatively more detail the progress that has and is being made in this direction.

11.1 LKFT in QED4

In massless QED4 the LKFT imply that the fermion propagator is multiplicatively renormalizable, i.e., $F(p) = A(p^2)^\nu$. Brown and Dorey, [84], argued that an arbitrary *ansatz* for the vertex does not satisfy the requirement of the multiplicative renormalizability (MR). It was realized that neither the bare vertex nor the BC-vertex were good enough to fulfill the the demands of MR.

- **Bare Vertex:**

  For the massless case, the bare vertex leads to the following SDE for the fermion propagator :

$$\frac{1}{F(p)} = 1 - \frac{\alpha\xi}{4\pi\nu}\left[\frac{2F(p)}{\nu+2} - F(\Lambda)\right] .$$

  This equation does not have any solution for A and ν, except in the Landau gauge where $A = 1$ and $\nu = 0$. Therefore, for the bare vertex, $F(p)$ has multiplicatively renormalizable solution only in the Landau gauge.

[6] For the two loop calculation of the fermion propagator, see for example [80].

- **Ball and Chiu:**

  If instead of the bare vertex, we would have used the BC-vertex, an analogous calculation would have led us to the following equation for $F(p)$ in the massless limit

$$\begin{aligned} \frac{1}{F(p)} &= 1 - \frac{\alpha\xi}{4\pi\nu}\, F(\Lambda) \\ &+ \frac{3\alpha}{16\pi}\, F(p) \left[\frac{5}{2} + 2\pi \cot\nu - \frac{1}{\nu} - \frac{2}{\nu+1} - \frac{1}{\nu+2} + \ln\frac{\Lambda^2}{p^2} \right] . \end{aligned}$$

  The explicit presence of the term $\ln\Lambda^2/p^2$ spoils the multiplicatively renormalizable solution.

- **Curtis and Pennington:**

  Curtis and Pennington, [28], realized that the transverse vertex is crucial in ensuring the multiplicative renormalizability of the fermion propagator. Usage of their proposal into the SDE for the fermion propagator renders the latter multiplicatively renormalizable giving

$$F(p) = \left[\frac{p^2}{\Lambda^2} \right]^{\nu} \tag{86}$$

  with $\nu = \alpha\xi/4\pi$.

- **Bashir and Pennington I:**

  This proposal also generates exactly the same fermion propagator as in eq. (86).

- **Bashir and Pennington II:**

  The generalization of the above ansatz still ensures Eq. (86).

- **Dong, Munczek and Roberts:**

  Dong, Munczek and Roberts imposed the so called transversality condition on the transverse vertex to make sure that the LKFT were satisfied both in 3 and 4 dimensions simultaneously. This vertex, however, exhibits logarithmic kinematic singularities in 3 dimensions.

- **Bashir, Kızılersü and Pennington:**

  It is easy to see that the solution $F(p) = (p^2/\Lambda^2)^{\nu}$, with $\nu = \alpha\xi/4\pi$ correctly embodies the leading log terms which are proportional to $\alpha\xi$, $(\alpha\xi)^2$, $(\alpha\xi)^3$, etc. as is obvious in the expansion $F(p) = 1 + \nu \ln p^2/\Lambda^2 + (\nu^2/2!)\ln^2 p^2/\Lambda^2 + \cdots$. Crucially, this is not the general solution, nor it is in agreement with perturbation theory beyond the leading logs. The general solution is $F(p) =$

$(p^2/\Lambda^2)^\gamma$ where γ is not zero in the Landau gauge. In fact, $\gamma = a\xi - \frac{3}{2}\,a^2 + \frac{3}{2}\,a^3 + \mathcal{O}(a^4)$, where $a = \alpha/4\pi$, [85]. The most general non perturbative construction of the transverse vertex required by the multiplicative renormalizability of the fermion propagator, replacing the exponent ν by γ was carried out by Bashir, Kızılersü and Pennington [27].

11.2 LKFT in QED3

In massless QED3 the LKFT imply that if we take the wavefunction renormalization to be 1 in the Landau gauge, then in an arbitrary covariant gauge [7]

$$F(p) \quad = \quad 1 - \frac{\alpha\xi}{2p} \tan^{-1}\left[\frac{2p}{\alpha\xi}\right] \,. \tag{87}$$

However, it has been shown that although this condition is satisfied at one loop order, it gets violated at the two loop order, leading to the following expression for the fermion propagator, [83]:

$$F(p) \quad = \quad 1 - \frac{\pi\,\alpha\xi}{4p} + \frac{\alpha^2\xi^2}{4p^2} - \frac{3\alpha^2}{4p^2}\left(\frac{7}{3} - \frac{\pi^2}{4}\right) + \mathcal{O}(\alpha^3)\,. \tag{88}$$

Now, starting from the value of this expression in the Landau gauge, LKFT yields

$$F(p) \quad = \quad 1 \; - \; \frac{\alpha\xi}{2p} \tan^{-1}\frac{2p}{\alpha\xi} \; - \; \frac{(28-3\pi^2)p^2\alpha^2}{(\alpha^2\xi^2+4p^2)^2} \,. \tag{89}$$

An *ansatz* for the vertex should yield this fermion propagator through the SDE, guaranteeing its correctness till the two loop order. Bare vertex or the Ball-Chiu vertex again do not come up to these expectations. The CP-vertex, BP-vertex and the DMG-vertex reproduce eq. (87) correctly but not eq. (89).

- **Burden and Roberts:**

  This vertex satisfies the WGTI but fails to comply with the requirements of the LKFT beyond the lowest order.

- **Burden and Tjiang:**

  The deconstruction of Burden and Tjiang reproduces eq. (87) correctly but not eq. (89).

- **Bashir and Raya:**

  By construction this vertex satisfies the LKFT at the one loop order and yields a fermion propagator which satisfies its LKFT at the two loop order.

[7] The gauge dependence of the infrared behaviour of this theory has also been studied in [86] with a different approach.

11.3 LKFT and the Massive Fermion Propagator

The discussion on the LKFT so far is limited to the cases when in the massless limit the vertex function results in the correctly behaved fermion propagator. To include the discussion of the massive fermions, we need to go beyond it. In [87], LKF transformed fermion propagator in massive QED3 and QED4 was obtained [8], starting from the simplest input which corresponds to the lowest order of perturbation theory, i.e., $S(p) = 1/(\not{p} - m_0)$ in the Landau gauge. On LKF transforming, the fermion propagator in an arbitrary covariant gauge was obtained. In the case of QED3, the result can be written in terms of basic functions of momenta. In QED4, the final expression is in the form of hypergeometric functions where coupling α enters as parameter of this transcendental function. Results are compared with the one loop expansion of the fermion propagator in QED4 and QED3, [42, 35], and perfect agreement is found up to terms independent of the gauge parameter at one loop, a difference permitted by the structure of the LKFT[9]. In QED3, the results are :

$$
\begin{aligned}
F(p) &= -\frac{\alpha\xi}{2p}\arctan\left[\frac{2p}{(2m_0+\alpha\xi)}\right] + \frac{2p(4p^2+\alpha^2\xi^2)}{\phi} \\
&\quad -\frac{\alpha\xi(4p^2+\alpha\xi(2m_0+\alpha\xi))}{\phi}\arctan\left[\frac{2p}{(2m_0+\alpha\xi)}\right], \qquad (90) \\
\mathcal{M}(p) &= \frac{8p^3 m_0}{\phi}, \qquad (91)
\end{aligned}
$$

where

$$
\phi = 2p(4p^2+\alpha\xi(2m_0+\alpha\xi)) - \alpha\xi(4p^2+(2m_0+\alpha\xi)^2)\arctan\left[\frac{2p}{(2m_0+\alpha\xi)}\right].
$$

In QED4, the corresponding expressions are

$$
\begin{aligned}
F(p) &= \frac{\Gamma(1+\nu)}{2m_0^2\,\Gamma(3+\nu)\;{}_2F_1\left(1+\nu,3+\nu;3;-\frac{p^2}{m_0^2}\right)}\left(\frac{m_0^2}{\Lambda^2}\right)^{-\nu} \\
&\quad\times\left[4m_0^2\Gamma^2(2+\nu)\;{}_2F_1^2\left(1+\nu,2+\nu;2;-\frac{p^2}{m_0^2}\right)\right. \\
&\quad\left.+p^2\,\Gamma^2(3+\nu)\,{}_2F_1^2\left(1+\nu,3+\nu;3;-\frac{p^2}{m_0^2}\right)\right], \qquad (92)
\end{aligned}
$$

[8] In the context of gauge technique, gauge covariance of the spectral functions in QED was studied in [77, 78, 79].

[9] The references for the mathematical details behind this exercise can be found in [88, 89, 90, 91, 92]

$$\mathcal{M}(p) = \frac{2m_0 \ {}_2F_1\left(1+\nu, 2+\nu; 2; -\frac{p^2}{m_0^2}\right)}{(2+\nu) \ {}_2F_1\left(1+\nu, 3+\nu; 3; -\frac{p^2}{m_0^2}\right)} , \tag{93}$$

with $\nu = \alpha\xi/(4\pi)$. These results match onto one loop perturbative results upto gauge parameter independent term as expected. In fact, these results should correctly reproduce all order terms in perturbation theory, of the type $(\alpha\xi)^n$, as pointed out in [25]. For the massless case, these results reduce to the well-known expressions, including the power-law for the wavefunction renormalization function, as expected.

11.4 LKFT and DCSB

As LKFT are non perturbative in nature, they would not only tell us how a perturbative Green function will transform with the variation of gauge but also how a non perturbative one would do so. The first exercise to our knowledge in this connection for the dynamically generated mass function has been carried out in [93] in the context of QED3. Thus knowing the solution of the SDE for the mass function in the Landau gauge, one can LKFT to any other gauge. One can parametrize the solution in the Landau gauge for practically any choice of the full vertex approximation with the following expressions[10] :

$$F(p,0) = F_0\theta(m_0 - p) + \theta(p - m_0) , \tag{94}$$

$$\mathcal{M}(p;0) = M_0\left[\theta(m_0 - p) + \frac{m_0^2}{p^2}\theta(p - m_0)\right] , \tag{95}$$

The wavefunction renormalization behaves like a constant (different from unity) for low momentum, and tends to one as $p \to \infty$. For the rainbow approximation $F_0 = 1$. The mass function is a constant for low momentum and falls off like $1/p^2$ for large momentum. Now we carry out the LKFT exercise. Consequently, in the large-p limit it is found that

$$\mathcal{M}(p;\xi) = \frac{C_3(\xi)}{p^2} + \mathcal{O}\left(\frac{1}{p^3}\right) , \tag{96}$$

$$F(p;\xi) = 1 + \mathcal{O}\left(\frac{1}{p}\right) , \tag{97}$$

where

$$C_3 = m_0^2 M_0 + \frac{4am_0(m_0^2 F_0 + 3M_0^2)}{3M_0\pi} ,$$

[10] In this section, we shall use the notation $F(p,\xi)$ and $\mathcal{M}(p,\xi)$ for the wavefunction renormalization and the mass function in the covariant gauge ξ.

with $a = \alpha\xi/2$ and in the low-p domain

$$\mathcal{M}(p;\xi) = \frac{C_1(\xi)}{C_2(\xi)} + \mathcal{O}(p^2) \,, \tag{98}$$

$$F(p;\xi) = -\frac{C_1^2(\xi)}{C_2(\xi)} - C_2(\xi)p^2 + \mathcal{O}(p^4) \,, \tag{99}$$

where

$$\begin{aligned} C_1(\xi) &= \frac{m_0 M_0}{a^4}\left(3m_0 - \frac{4a}{\pi}\right) + \frac{2am_0}{\pi(a^2+m_0^2)}\left(\frac{F_0}{M_0} - \frac{m_0^2 M_0}{a^4}\right) \\ &\quad -\frac{2}{\pi}\left(\frac{F_0}{M_0} + \frac{3m_0^2 M_0}{a^4}\right)\arctan\frac{m_0}{a} \\ C_2(\xi) &= \frac{2}{3M_0^2\pi}\left[\frac{am_0(3a^2F_0 + 5m_0^2F_0 - 2M_0^2)}{(a^2+m_0^2)^2} - 3F_0\arctan\frac{m_0}{a}\right]. \end{aligned}$$

One concludes from these expressions that qualitative features of the mass function and the wavefunction renormalization remain intact in any other gauge, i.e., both the functions behave in an identical fashion in the large and low-momentum regions as they do in the Landau gauge.

As the SDE of higher point Green functions are intrincately related to it, a meaningful truncation scheme which maintains the key features of the theory, such as gauge invariance, continues to be a challenging problem. Along with other guiding principles, a correct inclusion of the LKFT of the fermion propagator and the fermion-boson interaction is crucial for arriving at reliable conclusions. The above provides a preliminary study in this connection. Encouragingly, one finds expected behaviour of the fermion propagator for the asymptotic regimes of momenta for all ξ, a fact supported by earlier direct numerical solutions of the SDE (where no reference is made to the LKFT) in a small region of ξ close to the Landau gauge [36, 75]. The initial results for quenched QED3 presented here provide the starting point for a more rigorous and exact numerical study. In a recent work, such an exersice has been carried out [94]. There, the gauge dependence of the chiral condensate both in the quenched and unquenched versions of QED3 has been studied.

The need for such an exercise becomes all the more essential due to the findings reported in [37] for the chiral condensate. Although they employ a truncation scheme which preserves Ward identities, gauge dependence of the condensate is conspicuous specially for the unquenched case even in the immediate vicinity of the Landau gauge. It was pointed out in [94], that the origin of this unwanted behaviour owes itself to the fact that LKFTs were not satisfied, an essencial consequence of gauge invariance. Once these are incorporated, one obtains a practically gauge independent value of the chiral condensate for a very broad range of values of the covariant gauge parameer, Following is a representative figure for quenched QED3 to demonstrate the miraculous role played by the LKFT.

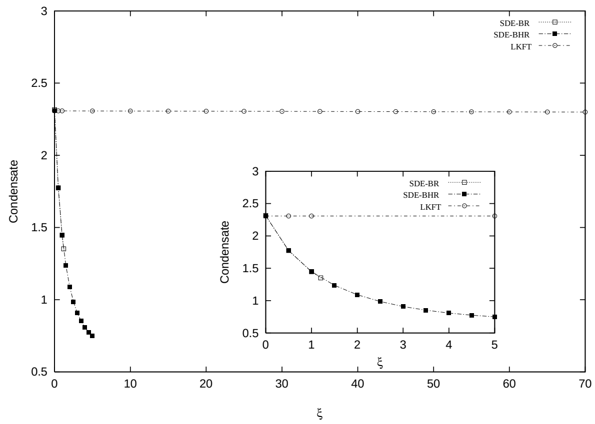


Figure 16. Gauge dependence of the chiral condensate in quenched QED3 employing the bare vertex.

One may wonder why we worry about the WGTI, kinematic singularitiesm CPT Symmetries and Perturbation Theory if, as suggested in [94], the LKFT are apparently sufficient to obtain the objective. The answer is that all the ingredientes are crucial in the hunt for a more reliable result in one gauge, i. e., the Landau gauge. This is where the choice of the full vertex plays an important role and the closer we are to the true vertex, the better it is. Once we know the result in the Landau gauge, LKFT will guide us along the path of varying the gauge. One can then go on to take up the more interesting case of QED4. The solution to this problem will in turn be the starting point for a study of QCD where the non-abelian nature of interactions, so essential for both confinement and asymptotic freedom, will further complicate the problem.

12 Conclusions

The non perturbative knowledge of the Green functions has vital importance in QCD when the coupling is not small enough to carry out a perturbative expansion and also in the dynamical models beyond the standard model of particle physics, such as the technicolour models, top condensate models, top color models etc. Schwinger-Dyson equations provide a natural tool for the continuum studies of all these scenarios. However, any non perturbative truncation scheme results in the loss of many deservedly sacred properties of these quantum field theories, including the gauge covariance of these Green functions resulting in the gauge dependent physical observables! Because of the highly involved structure of the Schwinger-Dyson equations, it is a difficult task to look for a truncation scheme which can solve this problem. However, a lot of progress is being made in simpler theories such as quenched QED in 3 and 4 dimensions to restore the gauge identities and render

the physical observables associated with the fermion propagator gauge independent. We have summarised the main attempts in this direction. Based upon these efforts, we hope that unquenching these theories and studying non-abelian theories such as QCD would become relatively less daunting and more accessible.
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